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The Structure of Cube Tilings Under Symmetry Conditions

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Abstract Let $m_1, \dots, m_d$ be positive integers, and let G be a subgroup of $\mathbb{Z}^d$ such that $m_1\mathbb{Z} \times \cdots \times m_d\mathbb{Z} \subseteq G$. It is easily seen that if a unit cube tiling $[0,1)^d + t$, $t \in T$, of $\mathbb{R}^d$ is invariant under the action of G, then for every $t \in T$, the number $|T \cap (t + \mathbb{Z}^d) \cap [0, m_1) \times \cdots \times [0, m_d)|$ is divisible by $|G|$. We give sufficient conditions under which this number is divisible by a multiple of $|G|$. Moreover, a relation between this result and the Minkowski–Hajós theorem on lattice cube tilings is discussed.

Keywords Cube tiling

1 Introduction

An interest in cube tilings of $\mathbb{R}^d$ originated from the following question raised by Hermann Minkowski [29]: Characterize lattices $\Lambda \subset \mathbb{R}^d$ such that $[0,1)^d + \lambda$, $\lambda \in \Lambda$, is a cube tiling. Minkowski conjectured that such a lattice Λ is of the form $A\mathbb{Z}^d$, where A is a lower triangular matrix for which every entry on the main diagonal equals 1. By a simple inductive argument it is equivalent to showing that Λ contains an element of the standard basis. Geometrically, it means that the tiling $[0,1)^d + \lambda$, $\lambda \in \Lambda$, contains a column. This inspired Ott-Heinrich Keller [16, 17] to consider the problem of the existence of columns in arbitrary (nonlattice) cube tilings. Eventually, Minkowski's conjecture has been confirmed by György Hajós [11], while Jeffrey Lagarias and Peter Shor [24] constructed a cube tiling without columns in dimension 10. Geometric

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and algebraic questions stemming from these problems attracted quite a number of researchers; see, e.g., [2, 3, 5, 6, 18, 20, 25–28, 30, 35].

Numerous authors pay their attention to tilings by clusters of cubes; see, e.g., [10, 12, 13, 21, 31–34]. Their investigations are rooted in coding theory and the Golomb–Welch conjecture [9]. These authors often deal with cubes which have their corners in $\mathbb{Z}^d$ and therefore the cluster tilings considered by them reduce to specific partitions of $\mathbb{Z}^d$. Other partitions of $\mathbb{Z}^d$ called disjoint covering systems are discussed in number theory; see, e.g., [1, 7].

A new stimulus to the theory of tilings came from Fuglede's conjecture [8]. A number of papers appeared at almost the same time where the set determining a cube tiling is characterized as follows: $[0,1)^d+t$, $t\in T$, is a cube tiling of $\mathbb{R}^d$ if and only if the system of functions $\exp(2\pi\,\mathrm{i}\langle t,x\rangle)$, $t\in T$, is an orthonormal basis of $L^2([0,1]^d)$ ([14, 15, 21, 23]).

A reader who seeks an exposition concerning cube tilings is advised to consult [22, 33, 36].

In this work we shall be concerned with cube tilings of $\mathbb{R}^d$ that are invariant under the action of certain groups of translations. Let $G\subset\mathbb{R}^d$ be a group. A cube tiling $[0,1)^d+t$, $t\in T$, is *G-invariant* if T is G-invariant; that is, $t+x\in T$ whenever $t\in T$ and $x\in G$. Let us remark that if T is a lattice, then the cube tiling is T-invariant. Observe that every G-invariant cube tiling leads to a tiling by clusters, where the union of all cubes $[0,1)+g$, $g\in G$, serves as a prototile.

Let $\boldsymbol{m}=(m_1,\dots,m_d)$ be a vector whose coordinates are positive integers, and let $\boldsymbol{e}_1,\dots,\boldsymbol{e}_d$ be the standard basis of $\mathbb{R}^d$. We say that the tiling $[0,1)^d+t$, $t\in T$, is *$\boldsymbol{m}$-periodic* if it is invariant under the action of the group generated by the vectors $m_i\boldsymbol{e}_i$, $i=1,\dots,d$. (If $\boldsymbol{m}$ is unspecified, then we simply say that the tiling is periodic.) Our main result reads as follows.

Theorem 1 *Let $\boldsymbol{m}$ be a vector whose coordinates are positive integers, and G be a subgroup of $\mathbb{Z}^d$ such that $m_1\mathbb{Z}\times\cdots\times m_d\mathbb{Z}\subseteq G$. Let $[0,1)^d+t$, $t\in T$, be a G-invariant cube tiling of $\mathbb{R}^d$. Let $G^i=G\cap\mathbb{Z}\boldsymbol{e}_i$, and $\mu_i=|\mathbb{Z}/G^i|$. Then for every $t\in T$, the number $|T\cap(t+\mathbb{Z}^d)\cap[0,m_1)\times\cdots\times[0,m_d)|$ is divisible by* $\mathrm{GCD}(\mu_1,\dots,\mu_d)|G|$.

We found it convenient to express our theorem in terms of cube tilings of flat tori.

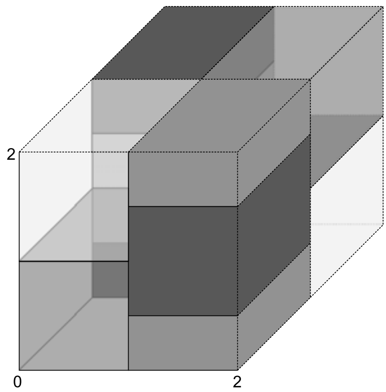
For a given $\boldsymbol{m}$, we define the *(flat) torus* $\mathbb{T}^d_{\boldsymbol{m}}$ as the set $[0,m_1)\times\cdots\times[0,m_d)$ with addition $\bmod\,\boldsymbol{m}$:

$$x\oplus y:=\bigl((x_1+y_1)\bmod m_1,\dots,(x_d+y_d)\bmod m_d\bigr).$$

Cubes in $\mathbb{T}^d_{\boldsymbol{m}}$ are the sets of the form $[0,1)^d\oplus t$, where $t\in\mathbb{T}^d_{\boldsymbol{m}}$. We say that $T\subset\mathbb{T}^d_{\boldsymbol{m}}$ *determines a cube tiling* of $\mathbb{T}^d_{\boldsymbol{m}}$ if the family $\mathcal{T}:=\{[0,1)^d\oplus t: t\in T\}$ is a *tiling*, that is, elements of $\mathcal{T}$ cover $\mathbb{T}^d_{\boldsymbol{m}}$ and are pairwise disjoint. Each $\boldsymbol{m}$-periodic cube tiling of $\mathbb{R}^d$ defines a cube tiling of $\mathbb{T}^d_{\boldsymbol{m}}$ in an obvious manner. Those sets which determine cube tilings of $\mathbb{T}^d_{\boldsymbol{m}}$ are characterized as follows:

Lemma 2 *T determines a cube tiling of $\mathbb{T}^d_{\boldsymbol{m}}$ if and only if*

Fig. 1 The G-invariant cube tiling of $\mathbb{T}^3_{(2,2,2)}$ determined by $T = \{(0, 0, 0), (0, 0, 1), (1, 1, 0), (1, 1, 1), (1, 0, 1/4), (1, 0, 5/4), (0, 1, 1/4), (0, 1, 5/4)\}$. The group G consists of two elements, $(0, 0, 0)$ and $(1, 1, 1)$



(1) $|T| = m_1 \times \cdots \times m_d$,
(2) *for every pair* $x, y \in \mathbb{T}_{\boldsymbol{m}}$, *there is* $i \in [d]$ *such that* $x_i - y_i$ *is a nonzero integer* (*Keller's condition*).

(Compare [36, Lemma 8.3].)

Let us mention that the notion of G-invariance carries over to cube tilings of $\mathbb{T}^d_{\boldsymbol{m}}$. The *integer lattice* in $\mathbb{T}^d_{\boldsymbol{m}}$ is the subset $\mathbb{Z}^d_{\boldsymbol{m}} := \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_d}$ of $\mathbb{T}^d_{\boldsymbol{m}}$ consisting of all vectors with integer coordinates. If T determines a cube tiling $\mathcal{T}$ of $\mathbb{T}^d_{\boldsymbol{m}}$ and $x \in T$, then the family $[0, 1)^d \oplus t$, $t \in T \cap (x \oplus \mathbb{Z}^d_{\boldsymbol{m}})$, is said to be a *simple component* of the tiling $\mathcal{T}$. In light of Lemma 2, we shall be mostly concerned with sets that determine cube tilings rather than cube tilings themselves. Therefore, we shall refer to $T \cap (x \oplus \mathbb{Z}^d_{\boldsymbol{m}})$ as a simple component of T. For $u \in [0, 1)^d$, let $C_u := T \cap (u \oplus \mathbb{Z}^d_{\boldsymbol{m}})$. It is clear that if C_u is nonempty, then it is a simple component of T. Moreover, for each simple component C of T, there is a unique element $u \in [0, 1)^d$ such that $C = C_u$.

Theorem 1 can now be stated for the flat tori (Fig. 1).

Theorem 3 *Let* T *determine a cube tiling of* $\mathbb{T}^d_{\boldsymbol{m}}$. *Let* G *be a subgroup of* $\mathbb{Z}^d_{\boldsymbol{m}}$. *Let* $G^i = G \cap (\mathbb{Z}_{m_i} \boldsymbol{e}_i)$, *where* $\boldsymbol{e}_i$ *is the* i*th element of the standard basis, and* $\mu_i = m_i/|G^i|$. *If* T *is* G-*invariant, then each simple component of* T *has its cardinality divisible by* $\mathrm{GCD}(\mu_1, \ldots, \mu_d)|G|$.

2 Proof of the Main Result

We shall need the following:

Lemma 4 *Let* V *be a nonempty subset of* $[0, 1)^d$. *If for every* $v \in V$ *and every* $i \in [d]$, *there is* $y \in V$ *such that* $v_q = y_q$ *whenever* $q \neq i$ *and* $v_i \neq y_i$, *then for every* $v \in V$, *there is* $u \in V$ *such that* $v_r \neq u_r$ *whenever* $r \in [d]$.

Proof By our assumptions, one can construct a sequence v^j, $j = 0, \dots, d$, consisting of elements of V such that $v^0 = v$ and $v^j_q = v^{j-1}_q$ for $q \neq j$ and $v^j_j \neq v^{j-1}_j$ for $j \in [d]$. Now, it suffices to set $u = v^d$. □

Proof of Theorem 3 If the theorem were false, then there would exist a counterexample with minimal d. Clearly, $d > 1$, as for $d = 1$, the theorem is evidently true. Let $n = \mathrm{GCD}(\mu_1, \dots, \mu_d)|G|$, and let $V \subset [0, 1)^d$ consist of all v such that C_v is a simple component of T and $|C_v|$ is not divisible by n. As a hypothetical counterexample is under discussion, V is nonempty. We are going to show that V satisfies the assumptions of Lemma 4. We have to check only the case $i = d$, as for the remaining cases, the same reasoning applies. Let us define $s : \mathbb{T}^d_{\boldsymbol{m}} \to \mathbb{T}^d_{\boldsymbol{m}}$ by the formula $s(t) = (t_1, \dots, t_{d-1}, \lfloor t_d \rfloor)$. Let $S = s(T)$. By Lemma 2, s restricted to T is a one-to-one mapping, and S determines a cube tiling of $\mathbb{T}^d_{\boldsymbol{m}}$. Moreover, since G is a subgroup of $\mathbb{Z}^d_{\boldsymbol{m}}$, it follows easily from the definition of S that

$$S \oplus G = \{s \oplus g : s \in S,\ g \in G\} = S.$$

For $z \in [0, 1)^d$, let $B_z = S \cap (z \oplus \mathbb{Z}^d_{\boldsymbol{m}})$. If $x \in T \cap [0, 1)^d$, then $z = s(x) \in S \cap [0, 1)^d$, and B_z is a simple component of S. Let $\boldsymbol{m}' = (m_1, \dots, m_{d-1})$. Then, by the definition of S, there are sets $S_i \subset \mathbb{T}_{\boldsymbol{m}'}$, $i \in \mathbb{Z}_{m_d}$, such that

$$S = S_0 \times \{0\} \cup \dots \cup S_{m_d - 1} \times \{m_d - 1\}.$$

Since S determines a cube tiling, each of the sets S_i, $i \in \mathbb{Z}_{m_d}$, determines a cube tiling of $\mathbb{T}_{\boldsymbol{m}'}$. Let G' be the subgroup of $\mathbb{T}_{\boldsymbol{m}'}$ defined by the equation $G' \times \{0\} = \{g \in G : g_d = 0\}$. By the definition of S_i and the fact that $S \oplus G = S$ we have $S_i \oplus G' = S_i$. Let $G'^i = G' \cap (\mathbb{Z}_{m_i} \boldsymbol{e}_i)$ for $i \in [d-1]$. Clearly, $G'^i \cong G^i$. Let $z \in S \cap [0, 1)^d$. The component B_z decomposes in a similar way to S:

$$B_z = B_0 \times \{0\} \cup \dots \cup B_{m_d - 1} \times \{m_d - 1\}.$$

If B_i is nonempty, then it is a simple component of S_i. By induction, $|B_i|$ is divisible by $n' = \mathrm{GCD}(\mu_1, \dots, \mu_{d-1})|G'|$ for every $i \in \mathbb{Z}_{m_d}$. Let $H = \{g_d : g \in G\}$. Since B_z is G-invariant, we have $|B_i| = |B_j|$ for every pair $i, j \in \mathbb{Z}_{m_d}$ such that $i - j \in H$. Therefore, $|B_z|$ is divisible by $n'|H| = \mathrm{GCD}(\mu_1, \dots, \mu_{d-1})|G|$. A fortiori, $|B_z|$ is divisible by n. Let $z = s(v)$, where $v \in V$, and let $E_v = \{w \in [0, 1)^d \cap T : s(w) = z\}$. Obviously B_z is a disjoint union of the sets $s(C_w)$, $w \in E_v$. Therefore, $|B_z| = \sum_{w \in E_v} |C_w|$. Since $v \in E_v$ and the number n divides $|B_z|$, and does not divide $|C_v|$, it follows that there is $y \in E_v \setminus \{v\}$ such that n does not divide $|C_y|$ as well. Thus, y belongs to V, and since $s(v) = s(y)$, we have $v_q = y_q$ whenever $q \neq i (= d)$. It means that V satisfies the assumptions of Lemma 4 as expected. Consequently, there are two elements v and u in V such that $v_r \neq u_r$ whenever $r \in [d]$. Let $t \in C_v$ and $t' \in C_u$. Then $t_j - t'_j$ is not an integer for any $j \in [d]$, which violates Keller's condition (Lemma 2). □

3 Remarks

A nonnegative integer n is *representable* by $\boldsymbol{m} \in \mathbb{N}^d$ if there are nonnegative integers $n_1, \dots, n_d$ such that

$$n = n_1 m_1 + \cdots + n_d m_d.$$

We have proved the following theorem [19], appealing to the result of Coopersmith and Steinberger on cyclotomic arrays [4]:

Theorem 5 *If T determines a cube tiling of a torus $\mathbb{T}^d_{\boldsymbol{m}}$ and C is a simple component of T, then $|C|$ is $\boldsymbol{m}$-representable.*

It is a weakness of Theorem 3 that it is not a generalization of Theorem 5. The conjectural generalization should be the following:

Let T determine a cube tiling of $\mathbb{T}^d_{\boldsymbol{m}}$. Let G be a subgroup of $\mathbb{Z}^d_{\boldsymbol{m}}$. Let $G^i = G \cap (\mathbb{Z}_{m_i} \boldsymbol{e}_i)$, where $\boldsymbol{e}_i$ is the ith element of the standard basis, and $\mu_i = m_i/|G^i|$. If T is G-invariant, then each simple component of T has the cardinality representable by $(\mu_1|G|, \dots, \mu_d|G|)$.

Such a generalization would lead, for example, to a new proof of the Minkowski–Hajós theorem on lattice cube tilings. We need less for this particular purpose. It would suffice to prove that, under the assumptions of Theorem 3, if all the numbers $\mu_1, \dots, \mu_d$ are greater than 1, then at least one of the simple components of T has cardinality greater than $|G|$. Theorem 3, as it is, suffices to show a restricted p-adic version of the Minkowski–Hajós theorem:

Let $\Lambda \subset \mathbb{R}^d$ be a cube tiling lattice. Suppose that there are a prime p and a positive integer k such that $p^k \boldsymbol{e}_i \in \Lambda$ for each $i \in [d]$. Then there is $j \in [d]$ such that $\boldsymbol{e}_j \in \Lambda$.

Indeed, let $T = \{\lambda \bmod \boldsymbol{m} \colon \lambda \in \Lambda\}$, where $\boldsymbol{m} = (p^k, \dots, p^k)$. By the fact that $p^k \boldsymbol{e}_i \in \Lambda$ for $i \in [d]$, it follows that T determines a cube tiling of $\mathbb{T}^d_{\boldsymbol{m}}$. Since Λ is a subgroup of $\mathbb{R}^d$, T is a subgroup of $\mathbb{T}^d_{\boldsymbol{m}}$. Let $G = \mathbb{Z}^d_{\boldsymbol{m}} \cap T$. Then G is a nonempty subgroup of T, as $0 \in G$. Therefore, T is G-invariant. By Theorem 3, the number $|\mathbb{Z}^d_{\boldsymbol{m}} \cap T| = |G|$ is divisible by $\mathrm{GCD}(\mu_1, \dots, \mu_d)|G|$, which, by taking into account that all μ_i are powers of p, readily implies that there is j such that $\mu_j = 1$. Thus, $p^k = |G^j|$, and consequently, $\boldsymbol{e}_j \in G^j \subseteq T$.

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